

# Circle theorems

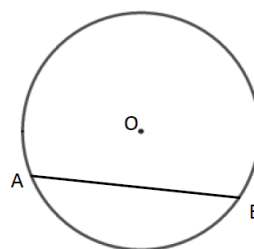
## A LEVEL LINKS

**Scheme of work:** 2b. Circles – equation of a circle, geometric problems on a grid

## Key points

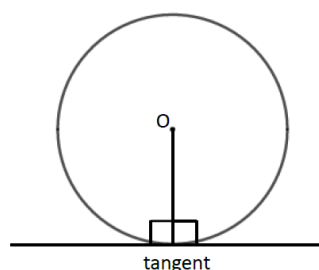
A chord is a straight line joining two points on the circumference of a circle.

So AB is a chord.



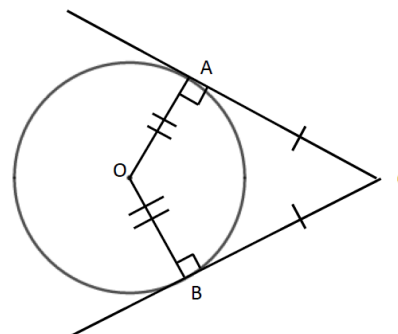
A tangent is a straight line that touches the circumference of a circle at only one point.

The angle between a tangent and the radius is  $90^\circ$ .



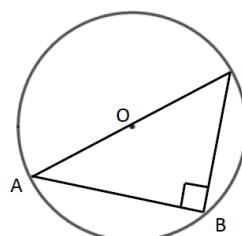
Two tangents on a circle that meet at a point outside the circle are equal in length.

So  $AC = BC$ .



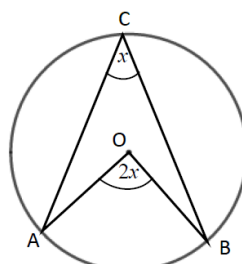
The angle in a semicircle is a right angle.

So angle  $ABC = 90^\circ$ .

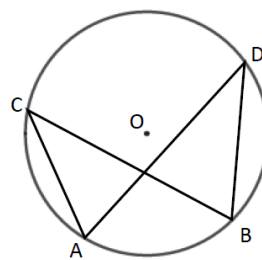


When two angles are subtended by the same arc, the angle at the centre of a circle is twice the angle at the circumference.

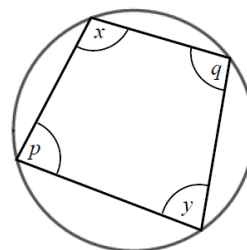
So angle  $AOB = 2 \times \text{angle } ACB$ .



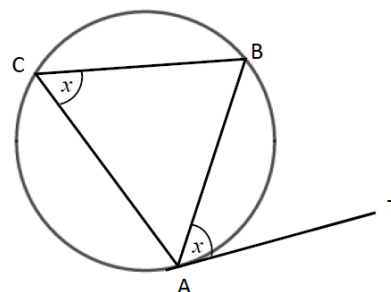
Angles subtended by the same arc at the circumference are equal. This means that angles in the same segment are equal.  
So angle  $ACB = \text{angle } ADB$  and angle  $CAD = \text{angle } CBD$ .



A cyclic quadrilateral is a quadrilateral with all four vertices on the circumference of a circle.  
Opposite angles in a cyclic quadrilateral total  $180^\circ$ .  
So  $x + y = 180^\circ$  and  $p + q = 180^\circ$ .

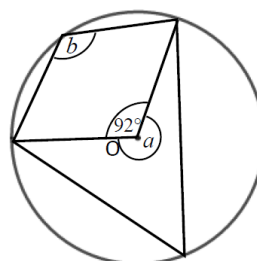


The angle between a tangent and chord is equal to the angle in the alternate segment, this is known as the alternate segment theorem.  
So angle  $BAT = \text{angle } ACB$ .



## Examples

**Example 1** Work out the size of each angle marked with a letter.  
Give reasons for your answers.

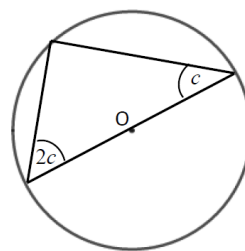


Angle  $a = 360^\circ - 92^\circ$   
 $= 268^\circ$   
as the angles in a full turn total  $360^\circ$ .

Angle  $b = 268^\circ \div 2$   
 $= 134^\circ$   
as when two angles are subtended by the same arc, the angle at the centre of a circle is twice the angle at the circumference.

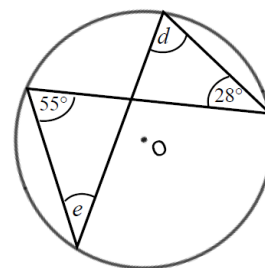
- 1 The angles in a full turn total  $360^\circ$ .
- 2 Angles  $a$  and  $b$  are subtended by the same arc, so angle  $b$  is half of angle  $a$ .

**Example 2** Work out the size of the angles in the triangle.  
Give reasons for your answers.



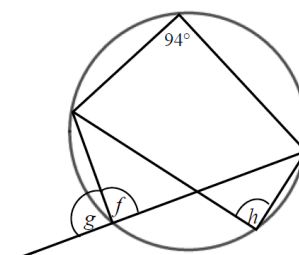
<p>Angles are <math>90^\circ</math>, <math>2c</math> and <math>c</math>.</p> $90^\circ + 2c + c = 180^\circ$ $90^\circ + 3c = 180^\circ$ $3c = 90^\circ$ $c = 30^\circ$ $2c = 60^\circ$ <p>The angles are <math>30^\circ</math>, <math>60^\circ</math> and <math>90^\circ</math> as the angle in a semi-circle is a right angle and the angles in a triangle total <math>180^\circ</math>.</p>	<ol style="list-style-type: none"> <li>1 The angle in a semicircle is a right angle.</li> <li>2 Angles in a triangle total <math>180^\circ</math>.</li> <li>3 Simplify and solve the equation.</li> </ol>
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**Example 3** Work out the size of each angle marked with a letter.  
Give reasons for your answers.



<p>Angle <math>d = 55^\circ</math> as angles subtended by the same arc are equal.</p> <p>Angle <math>e = 28^\circ</math> as angles subtended by the same arc are equal.</p>	<ol style="list-style-type: none"> <li>1 Angles subtended by the same arc are equal so angle <math>55^\circ</math> and angle <math>d</math> are equal.</li> <li>2 Angles subtended by the same arc are equal so angle <math>28^\circ</math> and angle <math>e</math> are equal.</li> </ol>
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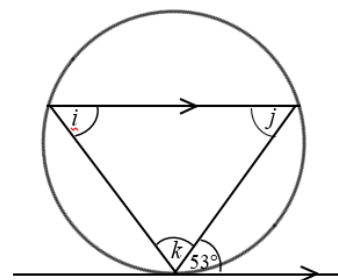
**Example 4** Work out the size of each angle marked with a letter.  
Give reasons for your answers.



<p>Angle <math>f = 180^\circ - 94^\circ</math> <math>= 86^\circ</math> as opposite angles in a cyclic quadrilateral total <math>180^\circ</math>.</p>	<ol style="list-style-type: none"> <li>1 Opposite angles in a cyclic quadrilateral total <math>180^\circ</math> so angle <math>94^\circ</math> and angle <math>f</math> total <math>180^\circ</math>.</li> </ol> <p>(continued on next page)</p>
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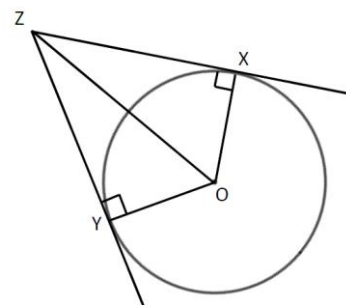
<p>Angle <math>g = 180^\circ - 86^\circ</math>  <math>= 84^\circ</math>  as angles on a straight line total <math>180^\circ</math>.</p> <p>Angle <math>h = \text{angle } f = 86^\circ</math> as angles subtended by the same arc are equal.</p>	<p><b>2</b> Angles on a straight line total <math>180^\circ</math> so angle <math>f</math> and angle <math>g</math> total <math>180^\circ</math>.</p> <p><b>3</b> Angles subtended by the same arc are equal so angle <math>f</math> and angle <math>h</math> are equal.</p>
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**Example 5** Work out the size of each angle marked with a letter. Give reasons for your answers.



<p>Angle <math>i = 53^\circ</math> because of the alternate segment theorem.</p> <p>Angle <math>j = 53^\circ</math> because it is the alternate angle to <math>53^\circ</math>.</p> <p>Angle <math>k = 180^\circ - 53^\circ - 53^\circ</math>  <math>= 74^\circ</math>  as angles in a triangle total <math>180^\circ</math>.</p>	<p><b>1</b> The angle between a tangent and chord is equal to the angle in the alternate segment.</p> <p><b>2</b> As there are two parallel lines, angle <math>53^\circ</math> is equal to angle <math>j</math> because they are alternate angles.</p> <p><b>3</b> The angles in a triangle total <math>180^\circ</math>, so <math>i + j + k = 180^\circ</math>.</p>
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**Example 6** XZ and YZ are two tangents to a circle with centre O. Prove that triangles XZO and YZO are congruent.



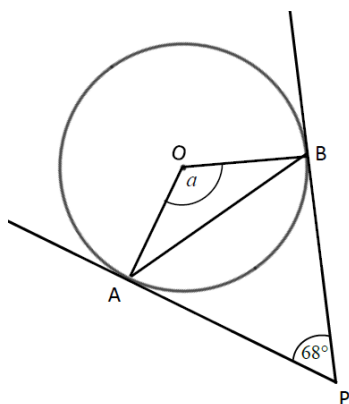
<p>Angle <math>OXZ = 90^\circ</math> and angle <math>OYZ = 90^\circ</math> as the angles in a semicircle are right angles.</p> <p>OZ is a common line and is the hypotenuse in both triangles.</p> <p><math>OX = OY</math> as they are radii of the same circle.</p> <p>So triangles XZO and YZO are congruent, RHS.</p>	<p>For two triangles to be congruent you need to show one of the following.</p> <ul style="list-style-type: none"> <li>• All three corresponding sides are equal (SSS).</li> <li>• Two corresponding sides and the included angle are equal (SAS).</li> <li>• One side and two corresponding angles are equal (ASA).</li> <li>• A right angle, hypotenuse and a shorter side are equal (RHS).</li> </ul>
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## Practice

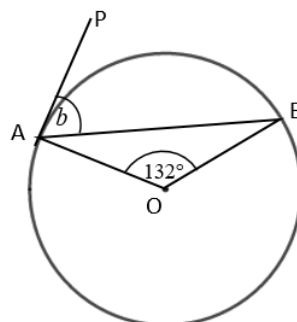
1 Work out the size of each angle marked with a letter.

Give reasons for your answers.

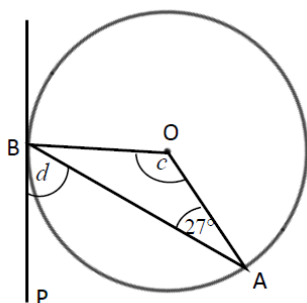
a



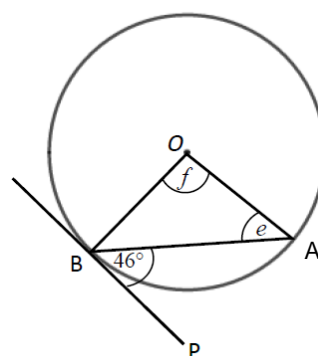
b



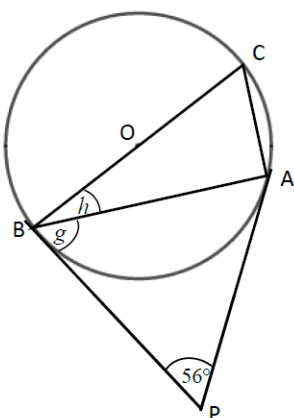
c



d



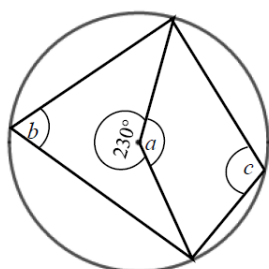
e



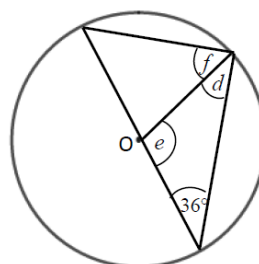
2 Work out the size of each angle marked with a letter.

Give reasons for your answers.

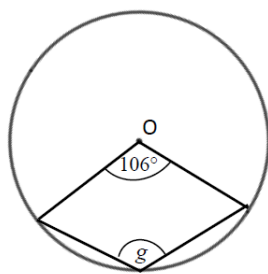
a



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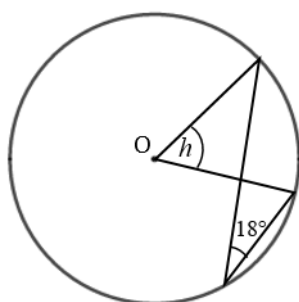
c



**Hint**

The reflex angle at point O and angle  $g$  are subtended by the same arc. So the reflex angle is twice the size of angle  $g$ .

d

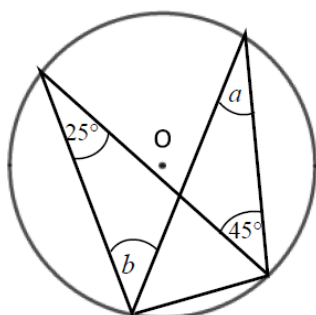


**Hint**

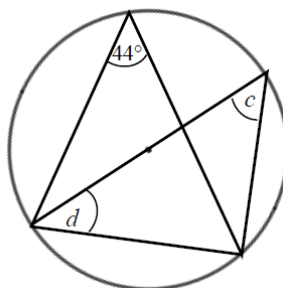
Angle  $18^\circ$  and angle  $h$  are subtended by the same arc.

- 3 Work out the size of each angle marked with a letter.  
Give reasons for your answers.

a



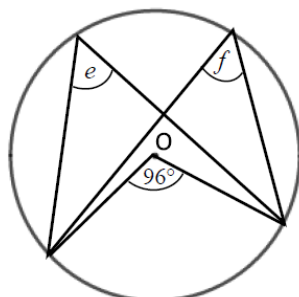
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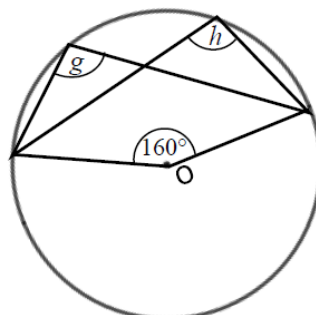
**Hint**

One of the angles is in a semicircle.

c

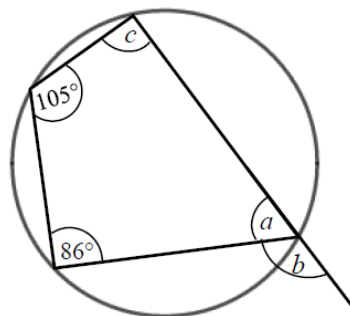


d



- 4 Work out the size of each angle marked with a letter.  
Give reasons for your answers.

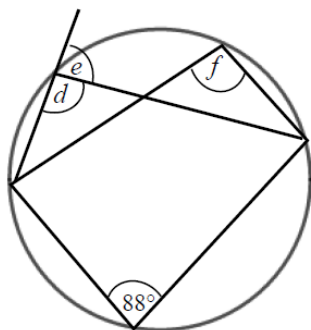
a



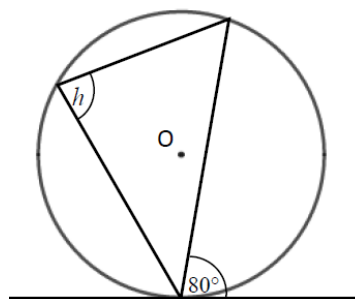
**Hint**

An exterior angle of a cyclic quadrilateral is equal to the opposite interior angle.

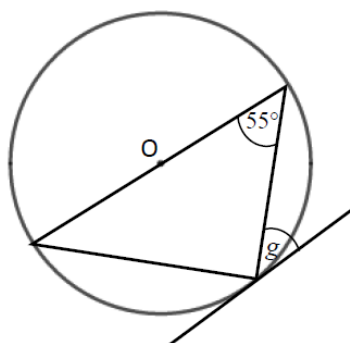
b



c



d



**Hint**

One of the angles is in a semicircle.

## Extend

- 5 Prove the alternate segment theorem.

## Answers

- 1
  - a  $a = 112^\circ$ , angle  $OAP = \text{angle } OBP = 90^\circ$  and angles in a quadrilateral total  $360^\circ$ .
  - b  $b = 66^\circ$ , triangle  $OAB$  is isosceles, Angle  $OAP = 90^\circ$  as  $AP$  is tangent to the circle.
  - c  $c = 126^\circ$ , triangle  $OAB$  is isosceles.  
 $d = 63^\circ$ , Angle  $OBP = 90^\circ$  as  $BP$  is tangent to the circle.
  - d  $e = 44^\circ$ , the triangle is isosceles, so angles  $e$  and angle  $OBA$  are equal. The angle  $OBP = 90^\circ$  as  $BP$  is tangent to the circle.  
 $f = 92^\circ$ , the triangle is isosceles.
  - e  $g = 62^\circ$ , triangle  $ABP$  is isosceles as  $AP$  and  $BP$  are both tangents to the circle.  
 $h = 28^\circ$ , the angle  $OBP = 90^\circ$ .
  
- 2
  - a  $a = 130^\circ$ , angles in a full turn total  $360^\circ$ .  
 $b = 65^\circ$ , the angle at the centre of a circle is twice the angle at the circumference.  
 $c = 115^\circ$ , opposite angles in a cyclic quadrilateral total  $180^\circ$ .
  - b  $d = 36^\circ$ , isosceles triangle.  
 $e = 108^\circ$ , angles in a triangle total  $180^\circ$ .  
 $f = 54^\circ$ , angle in a semicircle is  $90^\circ$ .
  - c  $g = 127^\circ$ , angles at a full turn total  $360^\circ$ , the angle at the centre of a circle is twice the angle at the circumference.
  - d  $h = 36^\circ$ , the angle at the centre of a circle is twice the angle at the circumference.
  
- 3
  - a  $a = 25^\circ$ , angles in the same segment are equal.  
 $b = 45^\circ$ , angles in the same segment are equal.
  - b  $c = 44^\circ$ , angles in the same segment are equal.  
 $d = 46^\circ$ , the angle in a semicircle is  $90^\circ$  and the angles in a triangle total  $180^\circ$ .
  - c  $e = 48^\circ$ , the angle at the centre of a circle is twice the angle at the circumference.  
 $f = 48^\circ$ , angles in the same segment are equal.
  - d  $g = 100^\circ$ , angles at a full turn total  $360^\circ$ , the angle at the centre of a circle is twice the angle at the circumference.  
 $h = 100^\circ$ , angles in the same segment are equal.
  
- 4
  - a  $a = 75^\circ$ , opposite angles in a cyclic quadrilateral total  $180^\circ$ .  
 $b = 105^\circ$ , angles on a straight line total  $180^\circ$ .  
 $c = 94^\circ$ , opposite angles in a cyclic quadrilateral total  $180^\circ$ .
  - b  $d = 92^\circ$ , opposite angles in a cyclic quadrilateral total  $180^\circ$ .  
 $e = 88^\circ$ , angles on a straight line total  $180^\circ$ .  
 $f = 92^\circ$ , angles in the same segment are equal.
  - c  $h = 80^\circ$ , alternate segment theorem.
  - d  $g = 35^\circ$ , alternate segment theorem and the angle in a semicircle is  $90^\circ$ .



5 Angle  $BAT = x$ .

Angle  $OAB = 90^\circ - x$  because the angle between the tangent and the radius is  $90^\circ$ .

$OA = OB$  because radii are equal.

Angle  $OAB = \text{angle } OBA$  because the base of isosceles triangles are equal.

Angle  $AOB = 180^\circ - (90^\circ - x) - (90^\circ - x) = 2x$  because angles in a triangle total  $180^\circ$ .

Angle  $ACB = 2x \div 2 = x$  because the angle at the centre is twice the angle at the circumference.

